Turán Problems for Mixed Graphs

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Background

Extremal Graph Theory

How large or small can a graph be, given that it satisfies certain structural constraints?

- A *graph* is a collection of vertices and edges.
- The *density* of a graph is $e/\binom{v}{2}$ where e, v are the number of edges and vertices.

Turán Problems

How dense can a graph be, given that it satisfies forbidden subgraph constraints?

Mixed Graphs

- *Mixed graph*: a graph with directed and undirected edges.
- *Mixed subgraphs*: obtainable by deleting vertices, deleting edges, and forgetting edge directions.

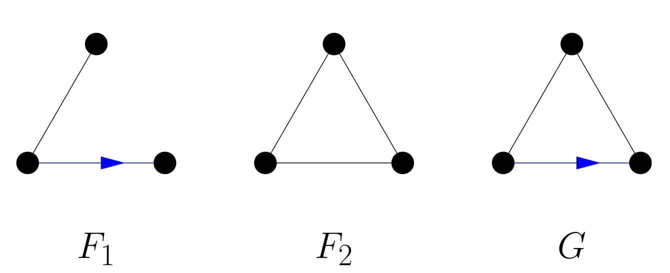

Figure 1. $F_1, F_2 \subseteq G$, and $F_1 \not\subseteq F_2$.

Classical Results

Turán's Theorem & the Erdős-Stone-Simonovits Theorem bound the density of a graph which does not contain some fixed subgraph.

Turán's Theorem [6]

The maximal density of a K_r-free graph is $\frac{r-2}{r-1}$.

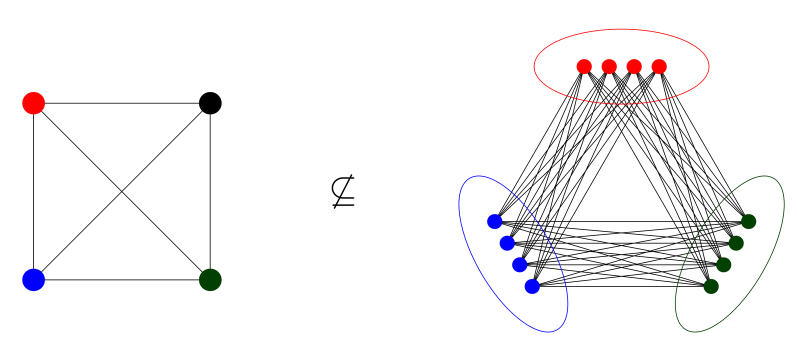

Figure 2. $K_4 \not\subseteq T(n, 3)$.

The Erdős-Stone-Simonovits Theorem [5, 4]

The Turán density of any graph F is

$$1 - \frac{1}{\chi(F) - 1},$$

where $\chi(F)$ is the *chromatic number* of F.

Abstract

We investigate natural Turán problems for mixed graphs, generalizations of graphs where edges can be either directed or undirected. We study a natural *Turán density coefficient* that measures how large a fraction of directed edges an F-free mixed graph can have; we establish an analogue of the Erdős-Stone-Simonovits theorem and give a variational characterization of the Turán density coefficient of any mixed graph, along with an associated extremal F-free family.

This characterization enables us to highlight an important divergence between classical extremal numbers and the Turán density coefficient. We show that Turán density coefficients can be irrational, but are always algebraic; for every positive integer k, we construct a family of mixed graphs whose Turán density coefficient has algebraic degree k.

Main Results

For mixed graph G on n vertices, let $\alpha(G)$ and $\beta(G)$ denote the undirected and directed edge densities. Let F be a mixed graph. The *Turán density coefficient* $\theta(F)$ is the maximum ρ such that

$$\alpha(G) + \rho\beta(G) \le 1 + o(1)$$

over all F-free n-vertex mixed graphs G. If $\beta(G) = o(1)$ over F-free n-vertex G, we say $\theta(F) = \infty$.

Theorem 1

Let F be a mixed graph. Then

- *$\theta(F) = 1$ iff F is uncollapsible (either two head vertices or two tail vertices are adjacent).*
- *$\theta(F) = \infty$ iff F admits a proper 2-coloring of the vertices with all head vertices the same color. Otherwise $\theta(F) \le 2$.*
- *If F has at most one directed edge, then*
$$\theta(F) = 1 + \frac{1}{\chi(F) - 2}.$$
- *Otherwise, we have that*
$$1 + \frac{1}{\chi(F)} \le \theta(F) \le 1 + \frac{1}{\chi(F) - 2}.$$

Theorem 2

Let F be a mixed graph with at least one directed edge where $\theta(F) \in (1, \infty)$. There is some family $\mathcal{B}_F$ of mixed graphs on at most $v(F)$ vertices, such that

$$\theta(F) = \min_{B \in \mathcal{B}_F} \left\{ \min_{\mathbf{y} \in \triangle^{v(B)-1}} \left\{ \frac{1 - \mathbf{y}^\intercal U_B \mathbf{y}}{\mathbf{y}^\intercal D_B \mathbf{y}} \right\} \right\},$$

where B is represented by (U_B, D_B), a pair of matrices encoding the undirected and directed edges of B, and $\triangle^{v(B)-1}$ is the simplex in $\mathbb{R}^{v(B)}$.

Theorem 3

The mixed graph in Figure 3 has irrational Turán density coefficient $\theta(F) = 1 + \frac{1}{\sqrt{2}}$.

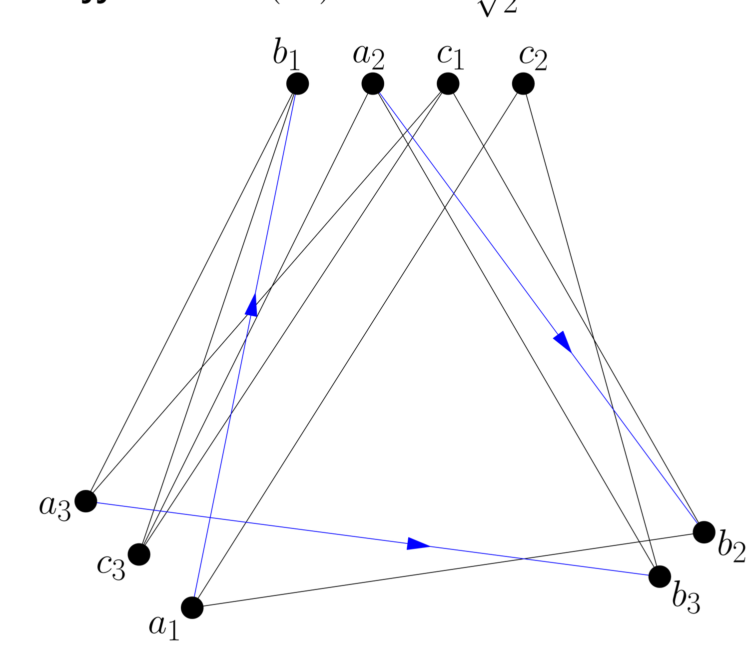

Figure 3. A mixed graph F with $\theta(F)$ irrational

Theorem 4

Let F be a mixed graph such that $\theta(F) < \infty$. Then $\theta(F)$ is an algebraic number.

Theorem 5

For every positive integer k, there exists a finite family of mixed graphs having Turán density coefficient with algebraic degree k.

Future Directions

- Is it possible to achieve all (or arbitrarily high) algebraic degrees with single mixed graphs F?
- What is the set of possible values of $\theta(F)$? (Do mixed graphs "jump"?)
- Generalize to *partially-directed hypergraphs*; applications to the k-SAT counting problem [1, 3].

Techniques

- **Classical extremal techniques**: Zykov symmetrization, Lagrangians, Turán-like constructions;
- **Probabilistic methods**: analogues of supersaturation, blowup results;
- **Combinatorial optimization**: quadratic forms, mixed adjacency matrices similar to those in Brown, Erdős, and Simonovits [2]

Applications

Mixed graphs are highly useful for encoding *relational information*:

- **Theoretical computer science**: extremal problems of coloring and job scheduling;
- **Neural networks**: object labeling and classification; Bayesian networks;
- **Other networks**: propositional logic; social network models; voting theory.

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